

On multiplicative congruences

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Abstract

Let ε be a fixed positive quantity, m be a large integer, x_j denote integer variables. We prove that for any positive integers N_1, N_2, N_3 with $N_1 N_2 N_3 > m^{1+\varepsilon}$, the set

$$\{x_1 x_2 x_3 \pmod{m} : x_j \in [1, N_j]\}$$

contains almost all the residue classes modulo m (i.e., its cardinality is equal to $m + o(m)$). We further show that if m is cubefree, then for any positive integers N_1, N_2, N_3, N_4 with $N_1 N_2 N_3 N_4 > m^{1+\varepsilon}$, the set

$$\{x_1 x_2 x_3 x_4 \pmod{m} : x_j \in [1, N_j]\}$$

also contains almost all the residue classes modulo m .

Let p be a large prime parameter and let $p > N > p^{63/76+\varepsilon}$. We prove that for any nonzero integer constant k and any integer $\lambda \not\equiv 0 \pmod{p}$ the congruence

$$p_1 p_2 (p_3 + k) \equiv \lambda \pmod{p}$$

admits $(1 + o(1))\pi(N)^3/p$ solutions in prime numbers $p_1, p_2, p_3 \leq N$.

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1 Introduction

In our works [6, 7] we applied large value results of character sums to a concrete multiplicative ternary congruence and by this mean improved one of the results of Friedlander and Shparlinski [5]. In the present paper we examine those arguments in application to some other multiplicative congruences.

Everywhere below ε denotes a small fixed positive quantity, m is a large integer parameter.

Theorem 1. *Let N_1, N_2, N_3 be positive integers with*

$$N_1 N_2 N_3 > m^{1+\varepsilon}.$$

Then for some $\delta = \delta(\varepsilon) > 0$ we have

$$\#\{x_1 x_2 x_3 \pmod{m} : x_j \in [1, N_j]\} = m + O(m^{1-\delta}).$$

In the statement of Theorem 1 the condition $N_1 N_2 N_3 > m^{1+\varepsilon}$ can not be relaxed to $N_1 N_2 N_3 > Cm$, no matter how large the constant C is. We also note that if $m = qn$, where q is a prime number approximately several times bigger than $n^{(1+\varepsilon)/(2-\varepsilon)}$, then $q > m^{(1+\varepsilon)/3}$ and hence non of the n numbers $q, 2q, \dots, nq$ can be represented in the form $x_1 x_2 x_3 \pmod{m}$ with $x_j \leq m^{(1+\varepsilon)/3}$. In particular the exponent of m inside of the O -symbol can not be replaced by a constant smaller than $(2 - \varepsilon)/3$.

It is known [8] that the set

$$\mathcal{A}_2 := \{x_1 x_2 \pmod{m} : x_1, x_2 \in [1, m^{1/2+\varepsilon}]\}$$

contains almost all the elements of the residue ring \mathbb{Z}_m . Theorem 1 implies that the set

$$\mathcal{A}_3 := \{x_1 x_2 x_3 \pmod{m} : x_1, x_2, x_3 \in [1, m^{1/3+\varepsilon}]\}$$

also contains almost all the elements of \mathbb{Z}_m . Another consequence of Theorem 1 is that for any sufficiently large integer m , any invertible element $\lambda \in \mathbb{Z}_m^*$ can be represented in the form

$$\lambda \equiv x_1 x_2 x_3 x_4 x_5 x_6 \pmod{m}$$

for some positive integers x_1, x_2, \dots, x_6 with $\max_{1 \leq i \leq 6} x_i \leq m^{1/3+\varepsilon}$.

Theorem 2. *Let m be cubefree, N_1, N_2, N_3, N_4 be positive integers with*

$$N_1 N_2 N_3 N_4 > m^{1+\varepsilon}.$$

Then for some $\delta = \delta(\varepsilon) > 0$ we have

$$\#\{x_1 x_2 x_3 x_4 \pmod{m} : x_j \in [1, N_j]\} = m + O(m^{1-\delta}).$$

In particular, for cubefree m the set

$$\mathcal{A}_4 := \{ x_1 x_2 x_3 x_4 \pmod{m} : x_1, x_2, x_3, x_4 \in [1, m^{1/4+\varepsilon}] \}$$

contains almost all the elements of \mathbb{Z}_m and also contains almost all the elements of \mathbb{Z}_m^* . This implies that, for any sufficiently large integer m , any element $\lambda \in \mathbb{Z}_m^*$ is representable in the form

$$\lambda \equiv x_1 x_2 x_3 x_4 x_5 x_6 x_7 x_8 \pmod{m}$$

for some positive integers x_1, x_2, \dots, x_8 with $\max_{1 \leq i \leq 8} x_i \leq m^{1/4+\varepsilon}$. It would be interesting to reduce the number of variables in the latter statement to 7.

Theorem 3. *Let p be a large prime parameter, k be a nonzero integer constant and λ be an integer coprime to p . If $p > N > p^{63/76+\varepsilon}$ then the congruence*

$$p_1 p_2 (p_3 + k) \equiv \lambda \pmod{p}$$

has $(1 + o(1))\pi(N)^3/p$ solutions in primes $p_1, p_2, p_3 \leq N$.

Theorem 3 quantitatively complements Theorem 6 from the work of Friedlander, Kurlberg and Shparlinski [4].

In what follows, the letters $\varepsilon', \varepsilon'', \varepsilon''', \varepsilon_1$ are used to denote some positive fixed quantities chosen in obvious ways. The letters x_j, y_j, u_j, t denote integer numbers.

2 Character sum estimates

In the proofs of Theorems 1, 2 we will use well-known character sum estimates of Burgess [1, 2]: if $N > m^{1/3+\varepsilon}$ then there exists $\delta = \delta(\varepsilon) > 0$ such that for any nonprincipal character $\chi \pmod{m}$ we have

$$\left| \sum_{x \leq N} \chi(x) \right| \leq N m^{-\delta}.$$

In the case when m is cubefree, the condition $N > m^{1/3+\varepsilon}$ can be relaxed to $N > m^{1/4+\varepsilon}$.

To prove Theorem 3 we shall use Vinogradov's bound on character sums over shifted primes. Let k be a fixed nonzero integer constant, χ be a nonprincipal character modulo p . Then Vinogradov's work [15] implies that in the range $1 \leq N < p$ one has

$$\left| \sum_{p' \leq N} \chi(p' + k) \right| \lesssim p^{1/4} N^{2/3}, \tag{1}$$

where p' denotes prime numbers. Here and below, we use the notation $L \lesssim M$ to indicate that for any fixed $\varepsilon > 0$ there exists a constant $c = c(\varepsilon)$ such that $L \leq cMp^\varepsilon$ (or $L \leq cMm^\varepsilon$ in the proofs of Theorems 1, 2).

The bound (1) is nontrivial when $N > p^{3/4+\varepsilon}$. We mention that Karatsuba's work [12] implies a nontrivial bound in the wider range $N > p^{1/2+\varepsilon}$. Since we deal with larger values of N , the estimate (1) will be more profitable.

3 Large values of character sums

Having character sum estimates under hands, one can apply Karatsuba's method from [13] to derive a variety of results on solvability of multiplicative ternary congruences and find asymptotic formulas for the number of their solutions. Our theorems, however, can not be obtained from the direct application of Karatsuba's method combined with Burgess' and Vinogradov's character sum estimates. One main ingredient in our proofs is Huxley's refinement of the Halász-Montgomery method for large value results of Dirichlet polynomials. Our present application of this theory can be compared with Lemma 4 of Friedlander and Iwaniec [3]. For our purposes it suffices the following simplest form of it. Let a_n be numbers with $|a_n| \lesssim 1$, let $0 < V \leq N$ and let R be the number of characters $\chi \pmod{p}$ for which

$$\left| \sum_{n=N+1}^{2N} a_n \chi(n) \right| \geq V.$$

Then Huxley's refinement implies that

$$R \lesssim \frac{N^2}{V^2} + \frac{pN^4}{V^6}, \quad (2)$$

see Montgomery [14], Huxley [9], Huxley and Jutila [10], Jutila [11].

The estimate (2) will be used in the proof of Theorem 3. A suitable version of it can also be used to prove Theorems 1, 2, but in this case we can present the proof in a relatively more elementary language, so that it will be more self-contained.

4 Proof of Theorem 1

It suffices to prove the following lemma:

Lemma 1. *Let $N_1 N_2 N_3 > m^{1+\varepsilon}$. Then there are only $O(m^{1-\varepsilon_1})$ elements $\lambda \in \mathbb{Z}_m^*$ such that*

$$\lambda \notin \{x_1 x_2 x_3 \pmod{m} : x_j \in [1, N_j]\}.$$

Indeed, assume that Lemma 1 is proved and we show how to derive Theorem 1 from this lemma.

In the condition of Theorem 1 we can assume that $N_1 > m^{(1+\varepsilon)/3}$. Denote by \mathcal{H} the set of all elements $\lambda \in \mathbb{Z}_m$ such that

$$\lambda \notin \{x_1 x_2 x_3 \pmod{m} : x_j \in [1, N_j]\}.$$

For a given divisor $d|m$, let \mathcal{H}_d be the set of all elements of \mathcal{H} such that $(h, m) = d$ for any $h \in \mathcal{H}_d$. Since $|\mathcal{H}_d| \leq m/d$, we have

$$|\mathcal{H}| = \sum_{d|m} |\mathcal{H}_d| = \sum_{\substack{d|m \\ d < m^{\varepsilon'}}} |\mathcal{H}_d| + \sum_{\substack{d|m \\ d \geq m^{\varepsilon'}}} m/d = \sum_{\substack{d|m \\ d < m^{\varepsilon'}}} |\mathcal{H}_d| + O(m^{1-0.5\varepsilon'}), \quad (3)$$

where $\varepsilon' = 0.1\varepsilon$ say. We estimate \mathcal{H}_d for $d < m^{\varepsilon'}$. By the definition,

$$\mathcal{H}_d = \{dx \pmod{m} : x \in \mathcal{B}_d\} \quad \text{for some } \mathcal{B}_d \subset Z_{m/d}^*.$$

Since

$$\mathcal{H}_d \cap \{x_1 x_2 x_3 \pmod{m} : x_j \in [1, N_j]\} = \emptyset,$$

taking $x_1 = dy_1$, $y_1 \in [1, N_1/d]$ we get that

$$\mathcal{B}_d \cap \{x_1 x_2 x_3 \pmod{(m/d)} : x_1 \in [1, N_1/d], x_2 \in [1, N_2], x_3 \in [1, N_3]\} = \emptyset.$$

Since $\mathcal{B}_d \subset Z_{m/d}^*$ and $(N_1/d)N_2N_3 > (m/d)^{1+0.5\varepsilon}$, we can apply Lemma 1 with m replaced by m/d and N_1 replaced by $[N_1/d]$ and deduce that

$$|\mathcal{H}_d| = |\mathcal{B}_d| = O(m^{1-\varepsilon''}), \quad \varepsilon'' > 0.$$

Incorporating this into (3), we conclude that

$$|\mathcal{H}| = O(m^{1-\varepsilon'''}), \quad \varepsilon''' > 0.$$

Thus, it suffices to prove Lemma 1. We can assume that $m^{0.1\varepsilon} < N_j < m$ for all j . Indeed, if say $N_1 < m^{0.1\varepsilon}$, then $N_2N_3 > m^{1+0.9\varepsilon}$ and we simply can take $x_1 = 1$ and look for $x_2 = y_1y_2$ with $y_1, y_2 \in [1, N_2^{1/2}]$.

Substituting $x_1 \rightarrow ux_1$ and manipulating with ε it suffices to show that if $N_1N_2N_3 > m^{1+\varepsilon}$ then for some $\varepsilon_1 > 0$,

$$\#\{ux_1x_2x_3 \pmod{m} : u \in [1, U], x_j \in [1, N_j]\} = m + O(m^{1-\varepsilon_1}),$$

where $U = [m^{1/n}]$, $n = [10/\varepsilon]$. We can assume that $N_1 > m^{(1+\varepsilon)/3}$. From the Burgess character sum estimate, there exists a positive quantity $\delta = \delta(\varepsilon) > 0$ such that

$$\left| \sum_{x_1 \leq N_1} \chi(x_1) \right| \leq N_1 m^{-\delta}. \quad (4)$$

Let \mathcal{H} be the set of all elements of \mathbb{Z}_m^* such that for each $h \in \mathcal{H}$ the congruence

$$h \equiv ux_1x_2x_3 \pmod{m}, \quad u \in [1, U], \quad x_j \in [1, N_j]$$

is not solvable. Therefore, since $(h, m) = 1$, we have

$$\sum_{\chi} \sum_{u \leq U} \sum_{x_1 \leq N_1} \sum_{x_2 \leq N_2} \sum_{x_3 \leq N_3} \sum_{h \in \mathcal{H}} \chi(ux_1x_2x_3) \overline{\chi}(h) = 0.$$

Separating the term corresponding to the principal character $\chi = \chi_0$, we get that

$$UN_1N_2N_3|\mathcal{H}| \lesssim \sum_{\chi \neq \chi_0} \left| \sum_{u \leq U} \chi(u) \right| \left| \sum_{x_1, x_2, x_3} \chi(x_1x_2x_3) \right| \left| \sum_{h \in \mathcal{H}} \chi(h) \right|. \quad (5)$$

Here we used the fact that the intervals $[1, U]$ and $[1, N_j]$ contain accordingly $Um^{o(1)}$ and $N_jm^{o(1)}$ numbers coprime to m (consider, for example, the primes of these intervals that are not divisors of m).

The set of nonprincipal characters $\chi \pmod{m}$ we split into two subsets:

$$\begin{aligned} \mathcal{A} &:= \{ \chi \pmod{m} : \left| \sum_{u \leq U} \chi(u) \right| \geq Um^{-\delta/4n} \}, \\ \mathcal{B} &:= \{ \chi \pmod{m} : \left| \sum_{u \leq U} \chi(u) \right| < Um^{-\delta/4n} \}. \end{aligned}$$

It follows that

$$\frac{|\mathcal{A}|U^{2n}m^{-\delta/2}}{\varphi(m)} \leq \frac{1}{\varphi(m)} \sum_{\chi} \left| \sum_{u \leq U} \chi(u) \right|^{2n}. \quad (6)$$

The right hand side of this inequality is not greater than the number of solutions of the congruence

$$u_1u_2 \cdots u_n \equiv u_{n+1}u_{n+2} \cdots u_{2n} \pmod{m}, \quad u_j \in [1, U].$$

In view of $U^n \leq m$, this congruence implies the equality

$$u_1u_2 \cdots u_n = u_{n+1}u_{n+2} \cdots u_{2n}.$$

Since any positive integer x has $x^{o(1)}$ divisors, the number of solutions of this equation is $U^{n+o(1)}$. Thus, from (6) it follows that

$$\frac{|\mathcal{A}|U^{2n}m^{-\delta/2}}{m} \lesssim U^n.$$

Since $U^n \approx m$, we get that

$$|\mathcal{A}| \lesssim m^{\delta/2}.$$

Therefore, applying Burgess bound to the sum over x_1 , we obtain that

$$\begin{aligned} \sum_{\chi \in \mathcal{A}} \left| \sum_{u \leq U} \chi(u) \right| \left| \sum_{x_1 \leq N_1} \chi(x_1) \right| \left| \sum_{x_2 \leq N_2} \chi(x_2) \right| \left| \sum_{x_3 \leq N_3} \chi(x_3) \right| \left| \sum_{h \in \mathcal{H}} \chi(h) \right| &\lesssim \\ &\lesssim m^{\delta/2} U N_1 m^{-\delta} N_2 N_3 |\mathcal{H}| \lesssim m^{-\delta/2} U N_1 N_2 N_3 |\mathcal{H}|. \end{aligned}$$

Inserting this into the inequality (5), we see that the sum over $\chi \in \mathcal{A}$ never dominates, and we therefore get

$$U N_1 N_2 N_3 |\mathcal{H}| \lesssim \sum_{\chi \in \mathcal{B}} \left| \sum_{u \leq U} \chi(u) \right| \left| \sum_{x_1, x_2, x_3} \chi(x_1 x_2 x_3) \right| \left| \sum_{h \in \mathcal{H}} \chi(h) \right|.$$

The sum over u we estimate in accordance with the definition of the set \mathcal{B} . This implies, after cancelation by U ,

$$N_1 N_2 N_3 |\mathcal{H}| \lesssim m^{-\delta/4n} \sum_{\chi \in \mathcal{B}} \left| \sum_{x_1, x_2, x_3} \chi(x_1 x_2 x_3) \right| \left| \sum_{h \in \mathcal{H}} \chi(h) \right|.$$

Now extending the summation over $\chi \in \mathcal{B}$ to the set of all characters $\chi \pmod{m}$ and then applying the Cauchy-Schwarz inequality, we deduce

$$\begin{aligned} \sum_{\chi \in \mathcal{B}} \left| \sum_{x_1, x_2, x_3} \chi(x_1 x_2 x_3) \right| \left| \sum_{h \in \mathcal{H}} \chi(h) \right| &\leq \\ &\leq \left(\sum_{\chi} \left| \sum_{x_1, x_2, x_3} \chi(x_1 x_2 x_3) \right|^2 \right)^{1/2} \left(\sum_{\chi} \left| \sum_{h \in \mathcal{H}} \chi(h) \right|^2 \right)^{1/2} \leq \sqrt{m I m |\mathcal{H}|}, \end{aligned}$$

where I is the number of solutions of the congruence

$$x_1 x_2 x_3 \equiv y_1 y_2 y_3 \pmod{m}, \quad x_j, y_j \in [1, N_j]. \quad (7)$$

Thus,

$$N_1 N_2 N_3 |\mathcal{H}| \lesssim m^{-\delta/4n} \sqrt{m I m |\mathcal{H}|}. \quad (8)$$

Now we write the congruence (7) as the equation

$$x_1 x_2 x_3 = y_1 y_2 y_3 + m t, \quad x_j, y_j \in [1, N_j], \quad |t| \leq N_1 N_2 N_3 / m$$

and observe that if we fix the quadruple (y_1, y_2, y_3, t) with $t \geq 0$, then this equation will have $m^{o(1)}$ solutions in variables x_1, x_2, x_3 . Since $N_1 N_2 N_3 >$

$m^{1+\varepsilon}$, there are less than $2(N_1N_2N_3)^2m^{-1}$ collections of such quadruples. Therefore,

$$I \lesssim (N_1N_2N_3)^2m^{-1}.$$

Plugging this into (8), we obtain

$$|\mathcal{H}| \lesssim m^{-\delta/4n} \sqrt{m|\mathcal{H}|}.$$

This implies $|\mathcal{H}| \lesssim m^{1-\delta/2n}$ and finishes the proof of Theorem 1.

5 Proof of Theorem 2

The proof is the same as the one of Theorem 1, where Lemma 1 should be replaced with the following one:

Lemma 2. *Let $N_1N_2N_3N_4 > m^{1+\varepsilon}$. Then there are only $O(m^{1-\varepsilon_1})$ elements $\lambda \in \mathbb{Z}_m^*$ such that*

$$\lambda \notin \{x_1x_2x_3x_4 \pmod{m} : x_j \in [1, N_j]\}.$$

The proof of Lemma 2 follows the same lines as the proof of Lemma 1. Here one uses Burgess' character sum estimate over the interval of length $N_1 > m^{(1+\varepsilon)/4}$ (such an estimate is guaranteed by the fact that m is cubefree).

6 Proof of Theorem 3

We assume that ε is as small positive quantity as we need below. Let J be the number of solutions of the congruence

$$p_1p_2(p_3 + k) \equiv \lambda \pmod{p}, \quad p_1, p_2, p_3 \leq N.$$

Expressing J via character sum estimates and separating the contribution from the principal character we get, for some $\delta' > 0$, that

$$J = \left(1 + O(p^{-\delta'})\right) \frac{\pi(N)^3}{p} + Error,$$

where

$$|Error| \ll \frac{1}{p} \sum_{\chi \neq \chi_0} \left| \sum_{p_1 \leq N} \chi(p_1) \right|^2 \left| \sum_{p_3 \leq N} \chi(p_3 + k) \right|.$$

We can split the interval of summation over p_1 into subintervals of the form $(N_1, N'_1]$, where $N_1 < N'_1 \leq 2N_1 < 2N$. Then decomposing into level sets, we get

$$|Error| \ll \frac{1}{p} RV_1^2 V_2 (\log q)^3, \quad (9)$$

where R is the number of non-principal characters χ for which

$$V_1 \leq \left| \sum_{p_1 \sim N_1} \chi(p_1) \right| \leq 2V_1, \quad V_2 \leq \left| \sum_{p_3 \leq N} \chi(p_3 + k) \right| \leq 2V_2.$$

If $V_1 \leq p^{\frac{5}{16}} N^{\frac{5}{12} + 0.01\varepsilon}$, then from (9) we get

$$|Error| \lesssim \left(\frac{RV_1^2}{p} \frac{RV_2^2}{p} \right)^{1/2} p^{\frac{5}{16}} N^{\frac{5}{12} + 0.01\varepsilon}.$$

Since

$$RV_1^2 \leq \sum_{\chi} \left| \sum_{p_1 \sim N_1} \chi(p_1) \right|^2 \leq pN_1, \quad RV_2^2 \leq \sum_{\chi} \left| \sum_{p_3 \leq N} \chi(p_3) \right|^2 \leq pN,$$

we get that

$$|Error| \lesssim Np^{5/16} N^{5/12 + 0.01\varepsilon}$$

and thus $Error = o(\pi(N)^3/p)$.

If $V_1 \geq p^{\frac{5}{16}} N^{\frac{5}{12} + 0.01\varepsilon}$, then in (9) we apply Vinogradov's bound (1) to get

$$|Error| \lesssim \frac{RV_1^2}{p} p^{1/4} N^{2/3}.$$

Then we use the large values estimate (2) to bound RV_1^2 :

$$RV_1^2 \lesssim N^2 + \frac{pN^4}{V^4} \lesssim \frac{pN^4}{p^{5/4 + 0.04\varepsilon} N^{5/3}}.$$

The result now follows.

7 Remarks

Theorems 1, 2 can be included into a more general statement. For instance, let k be fixed, N_1, N_2, \dots, N_k be positive integers such that $N_1 > m^{1/3 + \varepsilon}$ and $N_1 N_2 \cdots N_k > m^{1 + \varepsilon}$. Then the set

$$\{x_1 x_2 \cdots x_k \pmod{m} : x_j \in [1, N_j]\}$$

contains all, but $O(m^{1-\delta})$ elements of \mathbb{Z}_m . In case of cubefree m the condition $N_1 > m^{1/3+\varepsilon}$ can be replaced by $N_1 > m^{1/4+\varepsilon}$.

We can state Theorem 3 in the following form. Let $0 \leq \alpha < 1$, $0 \leq \beta < 1$ be fixed nonnegative real numbers. Define

$$\theta = \max\left\{\frac{\alpha}{1-\beta}, \frac{5+\alpha}{7-\beta}\right\}.$$

Let $p > N > p^{\theta+\varepsilon}$ and let for any nonprincipal character $\chi \pmod{p}$ we have

$$S_N \lesssim p^\alpha N^\beta.$$

Then the congruence

$$p_1 p_2 (p_3 + k) \equiv \lambda \pmod{p}$$

has $(1 + o(1))\pi(N)^3/p$ solutions in primes $p_1, p_2, p_3 \leq N$. The proof is the same as the proof of Theorem 3 (one considers the cases $V_1 \leq p^{\frac{1+\alpha}{4}} N^{\frac{1+\beta}{4}+0.01\varepsilon}$ and $V_1 \geq p^{\frac{1+\alpha}{4}} N^{\frac{1+\beta}{4}+0.01\varepsilon}$). In view of (1) the pair $(\alpha, \beta) = (1/4, 2/3)$ is acceptable, which produces $\theta = 63/76$. It would be interesting to obtain pairs (α, β) which would improve our exponent $63/76$.

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